The Origins of Four-Dimensional Geometry

In mathematician Felix Klein’s posthumously published memoir *Developments of Mathematics in the Nineteenth Century* (1926), Klein says of Hermann Grassmann that unlike “we academics who grow in strong competition with each other, like a tree in the midst of a forest which must stay slender and rise above the others simply to exist and to conquer its portion of light and air, he who stands alone can grow on all sides” (161). Grassmann never had a university position, taught only in German gymnasiums, and was consequently allowed to be a generalist: a philosopher, physicist, naturalist, and philologist who specialized in the Rig Veda, a Hindu classic. Grassmann’s mathematics was outside the mainstream of thought; read by few, his great work *Die lineale Ausdehnungslehre* (The Theory of Linear Extension, 1844) was described even by Klein as “almost unreadable.” Yet this book, more philosophy than mathematics, for the first time proposed a system whereby space and its geometric components and descriptions could be extrapolated to other dimensions.

Grassmann was not completely alone in his philosophical musings. August Möbius speculated that a left-handed crystal, structured like a left-turning circular staircase, could be turned into a right-handed crystal by passing it through a fourth dimension. Arthur Cayley published a paper on four-dimensional analytic geometry in 1844, at age twenty-two, and a few others worked on the idea of a general four-dimensional geometry. But these disparate musings lacked both a critical mass and a specific geometric interpretation.

In the second half of the nineteenth century, however, four-dimensional geometry advanced rapidly with the discovery and description of the four-dimensional analogs of the platonic solids, the geometric building blocks of space. In three dimensions there are five platonic solids: tetrahedron, cube, octahedron, icosahedron, and dodecahedron (fig. 1.1). They are “platonic” because they are regular: not only is every two-dimensional bounding face the same, but also each vertex is identical. In four dimensions, however, there are six platonic solids, also called polytopes (fig. 1.2).

According to the great Canadian geometer Harold Scott MacDonald Coxeter, the credit for the discovery of the platonic solids in four-dimensional space should go to Ludwig Schlafli (1814–1895). His book *Theorie der vielfachen Kontinuität* (Theory of Continuous Manifolds, 1852), with a title and a spirit so much like Grassmann’s but with an intensely analytic approach, went far beyond what had been done before. In calculus, an integral computes the area under a curve. By taking integrals of integrals of integrals, Schlafli computed the four-dimensional volumes of “poly-spheres.” Schlafli next extended Euler’s theory to
four dimensions. This remarkably useful formula, devised in the eighteenth century by Swiss mathematician Leonhard Euler, is often stated as follows: in a three-dimensional figure, the number of vertices minus the number of edges plus the number of faces minus the number of whole figures is equal to one \((v - e + f - c = 1)\). That is to say, in a cube the 8 vertices minus the 12 edges plus the 6 faces minus the 1 whole cube is equal to 1. Schlafli stated that the minus-plus, minus-plus pattern continues indefinitely beginning with vertices until the whole figure is reached. For a four-dimensional cube, or hypercube, Schlafli ultimately determined that the 16 vertices minus the 32 edges plus the 24 faces minus the 8 cubes, or cells, plus the 1 whole hypercube is equal to 1 \((v - e + f - c + u = 1)\). Knowing the Euler rule for regular figures, and being able to compute the volumes of higher-dimensional regular figures, Schlafli discovered which polytopes can fit inside which polyspheres, and also how to “dissect” the polytopes to reveal their lower-dimensional cells. Although Schlafli’s book was difficult, his results were clear and conclusive, and having solid objects to work with moved the effort from abstract and analytic to geometric and ultimately visual (box 1.1).

Thirty years later, in 1880, Washington Irving Stringham (1847–1909; box 1.2), a fellow of Johns
Schläfli’s results are elegantly summarized by what came to be called Schläfli symbols. The three-dimensional platonic solids are noted as {3, 3}, {4, 3}, {3, 4}, {3, 5}, and {5, 3} meaning, respectively, that three-sided faces fit three around each vertex to make a tetrahedron, four-sided squares are assembled three around each corner to make a cube, triangles fit four around each vertex to make an octahedron, and pentagonal faces meet three at a time to make a dodecahedron. The six regular, convex polytopes in four dimensions are then {3, 3, 3}, the four-dimensional tetrahedron, where three tetrahedra fit around each edge; {4, 3, 3}, the four-dimensional cube, or hypercube or tesseract, where three cubes fit around each edge making a total of 8 cells; {3, 3, 4} the four-dimensional octahedron, the 16-cell, where four tetrahedra fit around each edge; {3, 4, 3}, the four-dimensional cube-octahedron, which is regular in four dimensions although it is only semiregular in three, with 24 octahedral cells that fit three around each edge; {5, 3, 3}, the four-dimensional dodecahedron, or 120-cell, where three dodecahedra fit around each edge; and {3, 3, 5}, the four-dimensional icosahedron, the 600-cell, where five tetrahedra fit around each edge. Schläfli’s work also describes stellated versions of four of the ten polytopes discussed by Coxeter (pl. 1).
Lost to history until art historian Linda Henderson re-discovered his influential drawings of four-dimensional figures, Stringham remains a mostly unknown figure. Stringham was born 10 December 1847 in Yorkshire Centre (now Delavan) in western New York. Even then, when western New York was more populated, it was a bleak place: 100 miles from Buffalo, 100 miles from Erie, 100 miles from everywhere. As Calvin Moore recounts in his history of the mathematics department of the University of California at Berkeley, after the Civil War Stringham’s family moved to the relative sophistication of Topeka, Kansas. There, Stringham “established a house and sign painting business, and worked in a drugstore while attending Washburn College part-time. He also served as Librarian and teacher of penmanship at Washburn. With this unusual background, Stringham applied to and was admitted to Harvard College” (Moore, e-mail to the author, 4 February 2004).

Stringham received his bachelor’s degree from Harvard College in 1877 with highest honors. In 1878, he was granted admission to the graduate program at Johns Hopkins University. While reading his handwritten application to the mathematics department, I was pleased to note that he intended not only to study mathematics but also “as far as possible, to pursue the study of Fine Arts.” In his 20 May 1880 letter to the trustees of Johns Hopkins University in support of his request for a degree, Stringham listed ten courses of study during the previous year, including mainly calculus but also symbolic logic, quaternions, number theory, and physics. Last, he mentions, “I have been engaged privately in investigations in the Geometry of N. Dimensional Space.” In between January and May 1880, Stringham gave four talks on the subject to the Scientific Association and the Mathematical Seminary, the math club at Johns Hopkins started by William E. Story. These talks eventually became Stringham’s first paper in the American Journal of Mathematics. His study, seemingly unconnected to the degree program, may have been a continuation of work done the previous year listed as “other desultory work which I do not think worthy of mention” in a similar account to university president Daniel Coit Gilman (Gilman Papers, Milton S. Eisenhower Library, Johns Hopkins University).

After graduation from Johns Hopkins, Stringham went to Europe to see the sights and to study mathematics in Leipzig with the great German geometer Felix Klein (1849–1925). Stringham wrote Gilman with boyish excitement of his weekly seminars “with Prof. Klein’s wonderful critical faculty continually at play.” In the seminar, in addition to German students, there was “one Englishman, one Frenchman, one Italian, and one American (myself).” He hoped to negotiate a teaching position at Johns Hopkins or Harvard that would permit him to stay in Europe another year, but in the end Stringham reluctantly accepted the position of chair of the mathematics department at the University of California at Berkeley, starting in the fall of 1882. Stringham’s worst fears came true: though he soon entered the office of the dean and was the acting president of the college at the time of his sudden death in 1909, Stringham rarely had a chance to study modern mathematics or do any original work. In 1884, Stringham wrote to Gilman, who had previously been at Berkeley and who had gotten Stringham the job, about being bogged down in administrative affairs. He complained that the Board of Regents constantly intruded with an “arbitrary exercise of power in matters concerning which the judgement of the Faculty in Berkeley would certainly be more competent” and stated, “I have not been able to apply myself to my favorite studies.” Stringham published a few papers after this period, but mainly on the problems of teaching mathematics to undergraduate students. This wonderful mathematician with so many lively interests was swallowed by the morass that is university politics.

Yet Stringham likely took satisfaction in what he did accomplish during his years at Berkeley. When Stringham arrived the college had four hundred students and was a battleground between populist farmers and workers who saw the public university as a route to economic advancement, and patrician railroad barons who wanted it as a playground. Perhaps remembering his own modest beginnings, Stringham was committed to bringing a professional math curriculum to this public institution, and it remains a leading mathematical institution to this day.
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Fig. 1.3. Stringham’s exploded drawings of four-dimensional figures. Left to right: the four-dimensional tetrahedron, the hypercube, and the four-dimensional octahedron.

faces of the cube, which cuts off the cube's corners to result in a figure with 8 triangular faces, that is, two square-based pyramids joined on their bases. The four-dimensional 16-cell is generated by an analogous procedure that involves taking the centers of the 8 cells of the hypercube and joining these points with lines of equal length, making a compact figure of 24 edges, 32 faces, and 16 tetrahedral cells, 4 of which are wrapped around each edge. Again, Stringham asks us to imagine a folding up of pointed cells: “the edges of the figure are found by joining each summit with each of the other summits except its antipole, i.e. with six adjacent ones” (6).

Stringham constructed the hypercube in a different way, by extruding a cube into a fourth spatial dimension: “It may be generated by giving the 3-fold cube a motion of translation in the fourth dimension in a direction perpendicular to the three dimensional space in which it is situated. Each summit generated an edge, each edge a square, each square a cube” (5). Such a motion results in twice the number of vertices, edges, and faces of the original cube, as the “cube must be counted once for its initial and once for its final position.”

Like Schléfli, Stringham examined Euler’s formula extended to four dimensions. Stringham used the formula to discover the successive lower-dimensional cells of four-dimensional figures. Counting up the interior angles of the lower-dimensional cells, he could exclude four-dimensional figures whose lower-dimensional cells were too wide and too many to fit together without intersecting, even in four-dimensional space. By this method of counting their parts, Stringham rediscovered the possible arrangements of three-dimensional platonic solids as cells for four-dimensional figures. In the last section of his paper, Stringham extended this method to the fifth dimension, and he concluded, correctly, that only the tetrahedron, the cube, and the octahedron have analogs in five-dimensional space.

Stringham’s work was so influential (and seemingly unprecedented) at the time that it is a fair question to ask about his sources. Schléfli wrote his major work in 1852, but it was not published until 1901, six years after his death. As was common in the Journal at that time, Stringham gave no references, so it is unknown whether Stringham was familiar with Schléfli’s pioneering work or what other sources he may have had. There could, however, have been two distinct threads to Schléfli. Arthur Cayley translated large portions of Theorie der vielfachen Kontinuität and published them in 1858 and 1860 in Cambridge University’s Quarterly Journal of Pure and Applied Mathematics. Cayley changed the title to “On the Multiple Integral . . .” and treated this great work on the geometric n-dimensional polytopes as a treatise on problems in calculus, emphasizing the method rather than the results. Cayley’s translation would have been of interest to James Joseph Sylvester, who returned to America in 1876 to head the mathematics department at Johns Hopkins and became Stringham’s mentor there. Cayley and Sylvester were great friends; both were lawyers at the courts of Lincoln’s Inn in London, their day jobs for a time. Sylvester’s interest in a spatial fourth dimension was evident in an 1869 volume of Nature, in which he argued for “the
The development of technical or mechanical drawing is inextricably bound to the development of projective geometry, because both spring from Renaissance perspective. For their time, the techniques and texts of Filippo Brunelleschi (1377–1446), Leon Battista Alberti (1404–1472), and Piero della Francesca (ca. 1420–1492) represented both the most advanced geometry as well as the most advanced drawing techniques. The synergy provided by Renaissance technical drawing powered the scientific and technological progress of Europe.

Gaspard Monge (1746–1818), considered the inventor of modern technical drawing, continued the tradition of applying projective geometry to the description of useful objects. Monge’s work as an instructor in the military academy of Mézières and later as director of the École Polytechnique used projective geometry to design fortifications. This application was so important to Napoleon that it was kept secret until the publication of Monge’s *Geometria Descriptiva* in 1803. Monge’s basic technique was to project an object in space to a plane, then rotate that plane (with the image embedded) to lay flat on a page (fig. 1.4). Multiple projections further define the object of study. Monge’s most complicated drawing shows a cutaway drawing of the intersection of two cylinders (fig. 1.5). In Monge’s text, there are sections through objects but no exploded drawings that show how parts would fit when brought together. Victor Poncelet (1788–1867), Monge’s most original student, pondered his teacher’s work while a prisoner of war in Russia in 1813 and developed the purely mathematical side of projective geometry. Claude Crozet (1790–1864) brought the drawing techniques to the U.S. Military Academy at West Point, continuing the connection between the military and mechanical drawing.

William Minifie (1805–1888), an architect and teacher of drawing in Baltimore’s high schools, gave the discipline a tremendous boost with the publication of his *Text Book of Geometrical Drawing* in 1849. The book was used as a text throughout the United States and Great Britain. Even the first edition had a rather complete catalog of the techniques of mechanical drawing: geometric objects are shown with their “coverings” or as unfolded figures (fig. 1.6); with transparent faces or with parts removed; and as sections, elevations, and plans. Side and bottom views are shown rotated so that both lie adjacent on the page. Isometric and perspective views of objects are shown together. These are not just drawing techniques but tools and practices to develop visualization and conceptual understanding of the third dimension. By 1881, when Stringham had just published the first drawings of four-dimensional figures in the *American Journal of Mathematics*, Minifie’s book was in its eighth edition.

Stringham, as a teacher of penmanship and a professional sign painter, no doubt used Minifie’s widely accepted text. For his four-dimensional drawings, Stringham borrowed Minifie’s standard techniques. In particular, Stringham adapted the coverings of solids to depict his four-dimensional figures. However, there are no exploded drawings in Minifie’s work, which made Stringham’s use of them to show how three-dimensional cells fit together in a four-dimensional object all the more original. The exploded view did not come into
common usage until far into the twentieth century. Thomas Ewing French (1871–1944) inherited Minifie’s mantle as the professor of mechanical drawing, and neither his first edition of A Manual of Engineering Drawing (1911) nor the second edition of 1918 has an exploded drawing. They appear, in a most modest way, in the fifth edition of 1935 and are not fully exploited until much later.

Despite Stringham’s pioneering efforts, the full application of classical mechanical drawing techniques to four-dimensional figures is the work of the Dutch mathematician Pieter Hendrick Schoute (1846–1923).

Coming from a family of industrialists, Schoute received the best education Holland could provide, graduating as a civil engineer from the Polytechnic in Delft (now called the Technical University of Delft) in 1867. But young Hendrick did not want to be an engineer and instead pursued mathematics, receiving his doctorate from Leiden University in the Netherlands in 1870. For ten years, Schoute was forced to teach high school math before finally receiving a university appointment in Groningen, a city in a rural province in the north of Holland without much of a mathematics department in its university. Nevertheless, the secluded appointment gave Schoute a chance to sit down and seriously develop his interest in four-dimensional geometry, using the mechanical drawing techniques he had learned as an engineering student.

As later formalized in his Mehrdimensionale geometrie (1902), Schoute’s figures lay four mutually perpendicular axes of four-dimensional space—\(x_1, x_2, x_3, x_4\)—flat on the page (fig. 1.7). Line \(E\) is described as “half parallel” and “half normal [or perpendicular]” to both plane \(x_1, x_4\) and also plane \(x_2, x_3\), making those two planes absolutely perpendicular to each other, with only the point \(O\) in common, whereas plane \(x_1, x_4\), having an edge in common with plane \(x_1, x_2\), is therefore only partially perpendicular. There are actually six combinations of four axes, six planes of four-dimensional space that are mutually perpendicular at least to some degree. Because three views are sufficient in the mechanical drawing of civil engineering, however, Schoute apparently thought showing only four would suffice.

The Schoute formalism was adopted and extended by Esprit Jouffret. Given the history of mechanical drawing it is no surprise that he identified himself as an artillery lieutenant colonel and a former student of the École Polytechnique. Jouffret’s Traité élémentaire de géométrie à quatre dimensions (1903) shows...
Fig. 1.8. Drawings from Jouffret's 1903 and 1906 texts, respectively. Jouffret extended the Schoute technique of four-dimensional mechanical drawing.

Fig. 1.9. A drawing from Thorne's 1888 text on mechanical drawing. It is the earliest example found of the glass-box technique of mechanical drawing.

The four planes in the process of being unfolded and laid flat on the page (fig. 1.8A). A drawing from *Mélange de géométrie à quatre dimensions* (1906) shows all six planes of four-dimensional space passing through the origin (fig. 1.8B). This “glass box” approach is the fundamental gambit of mechanical drawing: possibly the first example of it appears in William Thorne’s *Junior Course: Mechanical Drawing* (1888; fig. 1.9).

By 1911 French’s work clearly articulated the glass-box metaphor. One imagined the object to be drawn inside a glass box with hinged faces. The image of the object was imprinted on the sides and top of the glass box. The box then opened flat with the views shown side by side. In the United States the convention is that the viewer is outside the box looking down on it, so that when the box is opened, the left view is to the left and the top view is on top, the so-called third-angle view. Most of the rest of the world uses the earlier first-angle view, where the viewer is inside the box with the object, showing the top view projected on the floor of the box beneath the viewer’s feet. Technical drawing is now primarily in the domain of computer graphics, and professors of architecture debate whether something is lost by replacing a pen with a mouse. Contemporary computer technical drawing of mathematical objects is the subject of chapter 10.
practical utility of handling space of four dimensions, as if it were conceivable space” (238).

However, Stringham’s visualizations and combinatorics are so different in style from Schläfli’s (and Sylvester’s) intense analysis that a different source is likely. In conversation, mathematician Dan Silver suggested to me a more likely thread connecting Schläfli and Stringham, one that runs via Johann Benedict Listing and William E. Story. When describing the omnianattentive outsider mathematician Grassmann, Felix Klein could have been describing Listing as well. Known as the father of topology and knot theory, Listing was most famous in his lifetime for his work on optics, and he was gifted in art and architecture. (Indeed, much of four-dimensional geometry was done by generalist mathematicians on the fringes of establishment thought.) In 1862, Listing published his Der Census räumlicher Complexe oder Verallgemeinerung des Eurler'schen Satzes von den Polyedern (The Census of Spatial Complexes or the Generalization of Euler’s Formula for Polyhedra). In Census, Listing followed Schläfli’s example and boosted Euler’s formula to four dimensions. It is likely that Listing’s visual approach, and his drawings in the back of the book, would have been appreciated by Stringham.

William Story (1850–1930) was a junior faculty member at Johns Hopkins during the 1880s and associate editor of the Journal. In the 1870s he had studied for his doctorate at the University of Leipzig, where Listing’s work would have been known. Story is the unsung hero of American four-dimensional geometry studies. Though he published little himself on the subject, his name turns up, behind the scenes, on many of the important nineteenth-century American papers. Indeed, in the only footnote to his paper, Stringham thanked Story for his help. Furthermore, in a letter defending himself against charges from a furious Sylvester of lateness and incompetence in editing the Journal, Story stated, “I worked this paper out very carefully with Stringham, giving him constantly suggestions and criticisms [because] Stringham had not [the paper] then in any kind of form” (Cooke and Rickey, 39).

Stringham’s method of visualizing the folding up of three-dimensional sections, or slices, to make four-dimensional figures extends the mechanical drawing techniques of his time. The folding visualization is easier to manage with figures made up of acute angles: the sharp-pointed “summits” of tetrahedra and octahedra, and the stellated versions that Stringham also drew for his illustration plates. It is harder to imagine cubic cells folding together at a point without distortion. Perhaps for this reason, Stringham seemed less at ease with the hypercube, which in some ways is the most logical of the four-dimensional solids because it is the easiest to imagine stacked into a Cartesian grid. He did draw the hypercube in projection, but he de-emphasized this figure in favor of the other figures. With his main efforts devoted to the exploded drawings of the three-dimensional covering cells, Stringham’s paper stops short of the projection model. The notion that in projection several spaces would be in the same place at the same time was alien to his thinking. In fact, such a phenomenon would be seen as evidence of error. As dazzling as it was at the time, Stringham’s taste for the solid assembly of parts was quite distinct from the modern taste for superimposition, multiplicity, and paradox.

On 7 July 1882, the German mathematician Victor Schlegel (1843–1905) presented a paper, “Quelque théorèmes de géométrie à n dimensions” (Some Theorems in n-Dimensional Geometry) to the Société Mathématique de France, which was published later that year in the society’s Bulletin. (Dutch mathematician Pieter Hendrick Schoute also presented a paper at this meeting.) The only reference cited by Schlegel was the Stringham work of 1880, but Schlegel presented a systematic discussion of the four-dimensional polytopes as projections, a topic barely mentioned by Stringham. Schlegel’s 1872 text System der Räumlehre (System of Spatial Theory) had demonstrated a
thorough understanding of projective geometry, and he was prepared to apply this discipline to four dimensions when the idea was introduced to him by Stringham's paper. Schlegel, yet another outsider, did far more than any other mathematician to establish the projection model. Although Schlegel received his doctorate from Leipzig—the prestigious crossroads for so many involved with this story—in 1881, when he was thirty-eight, he spent much of his career, both before and after earning his doctorate, as a teacher in vocational schools and gymnasiums, teaching mathematics and mechanical drawing.

Schlegel's choice of projection for a better representation of the four-dimensional figures is the origin of the more familiar Schlegel diagrams of three-dimensional forms that show all the faces of a polyhedron contained in a single face (for example, the look of a glass box to one pressing one's nose against a side). For the hypercube, "the most convenient is the following: one constructs a cube inside another, such that the faces of one are parallel (situées vis-à-vis) and one joins the vertices of one to the corresponding vertices of the other" (Schlegel 1882, 194). This is the hypercube drawn in four-dimensional perspective; there are four vanishing points (fig. 1.10). Schlegel does not say if such a perspective projection was original, nor does he indicate that the image was used elsewhere. Indeed, Schlegel chose an unusual viewpoint; he drew the hypercube from the point of view of one looking down from a corner. The purpose of such a drawing was to show that four lines of sight exist, one along each edge of the hypercube. Schlegel noticed that these lines of sight enclose a "pentaédroïde," or 5-cell, the four-dimensional simplex, thus demonstrating that the 5-cell has the same relation to the hypercube as the tetrahedron has to the cube. This insight led Schlegel to a general method for constructing the projection models of all the polytopes.

Only two years after Schlegel's perspective drawings of four-dimensional figures appeared in France, Schlegel built sculptural models of the polytopes and exhibited them in Halle, Germany. These models, made of thin metal rods and silk thread, were soon incorporated into the lively industry of mathematical model catalog sales from the late 1880s until at least the third decade of the twentieth century. Walter Dyck's catalog for an 1892 science museum exhibition in Munich lists metal wire and silk thread editions of the "projection models of the regular four-dimensional figures of Dr. V. Schlegel" as well as a projection model of the four-dimensional prism (fig. 1.11). Also listed were cardboard models of the interiors of the 120-cell and the 600-cell. Schlegel's models were sold through Brill, a mail-order house specializing in plaster casts of functions designed by mathematicians and manufactured to exacting standards. Included with any order of Schlegel's